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APPROXIMATE METHOD OF INTEGRATION OF LAMINAR BOUNDARY LAYER IN INCOMPRESSIBLE FLUID*

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Among all existing methods of the approximate integration of the differential equations of the laminar boundary layer, the most widely used is the method based on the application of the momentum equation (reference 1). The accuracy of this method depends on the more or less successful choice of a one-parameter family of velocity profiles. Thus, for example, the polynomial of the fourth degree proposed by Pohlhausen (reference 1) does not give velocity distributions closely agreeing with actual values in the neighborhood of the separation point, so that in the computations a strong retardation of the separation is obtained as compared with experimental results (reference 2). The more-accurate methods employed in recent times (references 2 to 4) assume as a single-parameter family of profiles the exact solutions of some special class of flows with given simple velocity distributions on the edge of the boundary layer (single term raised to a power, linear function).

The transition to the more complicated two- and more-parameter families of profiles would require, in addition to the momentum equation, the employment of other possible equations (for example, the equations of energy (reference 5) and others (reference 6)). A greater accuracy might also then be expected for relatively simple velocity profiles that satisfy only the fundamental boundary conditions on the surface of the body and on the edge of the boundary layer. This second approach, however, as far as is known, has not been considered except for very simple solution for the case of axial flow past a plate (reference 7).

In the present paper, a solution is given of the problem of the plane laminar boundary layer in an incompressible gas; the method is based on the use of a system of equations of successive moments (including that of zero moment, the momentum equation) of the equation of the boundary layer. Such statement of the problem

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leads to a complex system of equations, which, however, is easily solved for simple supplementary assumptions. The solution obtained is given in closed form by very simple formulas and is no-less accurate than the previously mentioned complicated solutions that are based on the use of special classes of accurate solutions of the boundary-layer equations.

1. Derivation of Fundamental System of Successive Moments of Boundary-Layer Equation. The well-known equations of the stationary plane laminar boundary layer in the absence of compressibility have the form

$$\left.\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= UU' + \nu\frac{\partial^2 u}{\partial y^2} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \end{aligned}\right\} \qquad (1.1)$$

where $u(x,y)$ and $v(x,y)$ are the projections of the velocity at a section of the boundary layer on the axial and transverse axes of coordinates x and y, $U(x)$ is a given longitudinal velocity on the outer boundary $U' = dU/dx$, and ν is the kinematic coefficient of viscosity. When the equation of continuity is applied, the first of equations (1.1) may be given the more convenient form

$$L(u,v) = \frac{\partial}{\partial x}\Big[u(U-u)\Big] + \frac{\partial}{\partial y}\Big[v(U-u)\Big] + U'(U-u) - \nu\frac{\partial^2(U-u)}{\partial y^2} = 0 \qquad (1.2)$$

The left side of equation (1.2) is multiplied by y^k and integrated with respect to y from zero to infinity in the case of an asymptotically infinite layer or from zero to the outer limit of the layer $y = \delta(x)$ for the assumption of a layer of finite thickness. In either case, the following expression is obtained:

$$\int_0^{\infty,\delta} L(u,v)y^k\,dy = \frac{d}{dx}\int_0^{\infty,\delta} y^k u(U-u)\,dy + \int_0^{\infty,\delta} y^k\frac{\partial}{\partial y}\Big[v(U-u)\Big]dy +$$

$$U'\int_0^{\infty,\delta} y^k(U-u)\,dy - \nu\int_0^{\infty,\delta} y^k\frac{\partial^2(U-u)}{\partial y^2}\,dy = 0 \qquad (1.3)$$

It is assumed in this equation and in what follows that, in view of the very rapid approach of the velocity difference $U - u$ to zero as $y \longrightarrow \infty$, all integrals with the infinite upper limit have a finite value.

For $k = 0$,

$$\frac{d}{dx} \int_0^{\infty, \delta} u(U-u)\, dy + U' \int_0^{\infty, \delta} (U-u)\, dy = \frac{\tau_w}{\rho} \tag{1.4}$$

where the magnitude

$$\tau_w = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0} \tag{1.5}$$

represents the friction stress on the surface of the body.

Equation (1.4), the well-known impulse or momentum equation, is readily transformed into its usual form

$$\frac{d\delta^{**}}{dx} + \frac{U'\delta^{**}}{U} (2+H) = \frac{\tau_w}{\rho U^2} \tag{1.6}$$

where

$$\left.\begin{aligned}
\delta^{*} &= \int_0^{\infty, \delta} \left(1 - \frac{u}{U}\right) dy \\
\delta^{**} &= \int_0^{\infty, \delta} \frac{u}{U} \left(1 - \frac{u}{U}\right) dy \\
H &= \frac{\delta^{*}}{\delta^{**}}
\end{aligned}\right\} \tag{1.7}$$

For $k = 1$, a new equation of the 'first moment' is obtained from equation (1.3)

$$\frac{d}{dx}\int_0^{\infty,\delta} yu(U-u)\ dy - \int_0^{\infty,\delta} v(U-u)\ dy + U'\int_0^{\infty,\delta} y(U-u)\ dy = \nu U \tag{1.8}$$

and, in general, for $k \geq 2$, the equations of successively increasing moments are obtained

$$\frac{d}{dx}\int_0^{\alpha,\delta} y^k u(U-u)\ dy - k\int_0^{\infty,\delta} y^{k-1}v(U-u)\ dy + U'\int_0^{\infty,\gamma} y^k(U-u)\ dy$$

$$= k(k-1)\nu\int_0^{\infty,\delta} y^{k-2}(U-u)\ dy \tag{1.9}$$

In all these equations, the transverse velocity $v(x,y)$ is assumed expressed in terms of the axial $u(x,y)$ from the equation of continuity.

It is now assumed that the family of functions

$$u = u^o\ (x,\ y;\lambda_1,\ \lambda_2,\ .\ .\ .\ ,\lambda_k) \tag{1.10}$$

satisfies the boundary conditions of the problem with k parameters $\lambda_1,\ .\ .\ .\ ,\ \lambda_k$, which are functions of x, such that the k successive moments of equation (1.3)

$$\int_0^{\infty,\delta} y^k L(u^o,\ v^o)\ dy \tag{1.11}$$

become zero. On the assumption that it is permissible to pass to the limit $k \to \infty$, it would then be possible to state that the function

$$u(x,y) = \lim_{k\to\infty} u^o\left[x,\ y;\ \lambda_1(x),\ \lambda_2(x),\ .\ .\ .\ ,\lambda_k(x)\right] \tag{1.12}$$

with parameters $\lambda_1(x), \lambda_2(x),\ .\ .\ .\ ,\lambda_k(x)$ satisfying the infinite system of equations

$$\int_0^{\infty, \delta} y^k L(u^o, v^o)\, dy = 0 \qquad (k = 0, 1, 2, \ldots)$$

or, what is equivalent, systems (1.4), (1.8), and (1.9) will be an exact solution of the fundamental system (1.1) for the assumed boundary conditions.

For this solution, it is merely necessary to recall the known theorem that a continuous function, all successive derivatives of which are equal to zero, is identically equal to zero (reference 8).

The question of the proof of the validity of this theorem is not considered in the case of an infinite interval or of an interval the boundaries of which are functions of a certain variable with respect to which the differentiation is effected. A certain construction, not based it is true on a rigorous proof, of the solution of the problem will be employed with the aid of the successive equations of the moments of the basic boundary-layer equation.

2. <u>Choice of Parameters of Family of Velocity Profiles at Sections of Boundary Layer. Special Form of Equations of Moments.</u> As is seen from the previously discussed considerations, the fundamental difficulty lies in the choice of a family of velocity profiles (1.10) and the determination of the parameters λ_k of the family. One of the simplest methods of the solution of this problem is indicated herein.

In the converging part of the boundary layer, the velocity profiles at various sections of the layer are known to be almost similar; the velocity profile is deformed mainly in the diffuser part of the boundary layer downstream of the point of minimum pressure. The deformation of the profile consists of the appearance of a point of flexure that arises near the surface of the body and then moves away from it as the separation point is approached.

The presence of this deformation of the profile near the surface should greatly affect the magnitude τ_w proportional to the normal derivative of the velocity on the surface of the body; it will therefore diminish to zero as the point of separation is approached. The deformation of the profile will have a smaller effect on such integral magnitudes as δ^* and δ^{**} and very little effect on magnitudes that contain under the integral sign functions that rapidly decrease as the surface of the body is approached.

For the parameters characterizing the effect of the deformation of the velocity profile, it is natural to assume those magnitudes that depend relatively strongly on the deformation of the velocity profile. With regard to the magnitudes that vary little with the deformation of the velocity profile, however, it is natural to assume that they do not depend on the chosen parameters.

For the fundamental parameters determining a change in the shape of the velocity profiles, which may be called form parameters, the nondimensional combination of the magnitudes τ_w, δ^* and δ^{**} will be employed with the given functions $U(x)$ and $U'(x)$ and physical constants, namely, the parameters

$$\left.\begin{aligned} f &= \frac{U'\delta^{**2}}{\nu} \\ \zeta &= \left(\frac{\partial(u/U)}{\partial(y/\delta^{**})}\right)_{y=0} = \frac{\tau_w\delta^{**}}{\mu U} \\ H &= \frac{\delta^*}{\delta^{**}} \end{aligned}\right\} \tag{2.1}$$

For the computation of the remaining magnitudes in the equation of moments according to the assumption, the velocity profile will be assumed in a section of the boundary layer in a form that does not depend on the parameters f, ζ, and H:

$$\frac{u}{U} = \varphi\left(\frac{y}{\delta^{**}}\right) = \varphi\ (\eta) \tag{2.2}$$

This assumption permits, as will be subsequently seen, obtaining on the basis of very simple computations a sufficiently accurate solution of the boundary-layer equations for arbitrary distribution of the velocity on the edge of the layer. The transformation of equation (1.6) will now be considered.

If the parameter ζ is introduced, then by equation (2.1),

$$\frac{d\delta^{**}}{dx} + \frac{U'\delta^{**}}{U}\ (2+H) = \frac{\nu}{U\ \delta^{**}}\zeta$$

or

$$\frac{U}{2}\frac{d}{dx}\left(\frac{\delta^{**2}}{\nu}\right) + (2+H)\ f = \zeta$$

It is not difficult to obtain finally

$$\frac{1}{2}\frac{U}{U'}\frac{df}{dx} + \left(2 + H - \frac{1}{2}\frac{UU''}{U'^2}\right)f = \zeta \tag{2.3}$$

For the transformation of the left side of equation (1.8), the first integral can be written by equation (2.2) $(\eta = y/\delta^{**})$

$$\int_0^\infty yu(U-u)\,dy = U^2\delta^{**2}\int_0^\infty \eta\varphi(1-\varphi)\,d\eta = H_1U^2\delta^{**2} \tag{2.4}$$

where the magnitude H_1, equal to

$$H_1 = \int_0^\infty \eta\varphi(1-\varphi)\,d\eta \tag{2.5}$$

represents a constant computed by the given function $\varphi(\eta)$.

In order to compute the following integral, the transverse velocity v is first expressed by the formula

$$v = -\int_0^y \frac{\partial u}{\partial x}\,dy = -\frac{\partial}{\partial x}\left(U\,\delta^{**}\int_0^\eta \varphi\,d\eta\right)$$

$$= -U'\delta^{**}\int_0^\eta \varphi d\eta - U\frac{d\delta^{**}}{dx}\int_0^\eta \varphi d\eta - U\,\delta^{**}\,\varphi\frac{d\eta}{dx}$$

or, when it is noted that

$$\frac{d\eta}{dx} = \frac{d}{dx}\left(\frac{y}{\delta^{**}}\right) = -\frac{y}{\delta^{**2}}\frac{d\delta^{**}}{dx} = -\eta\frac{1}{\delta^{**}}\frac{d\delta^{**}}{dx}$$

the following expression is obtained:

$$v = -U'\delta^{**}\int_0^\eta \varphi\,d\eta - U\frac{d\delta^{**}}{dx}\left(\int_0^\eta \varphi\,d\eta - \eta\varphi\right) \tag{2.6}$$

There is thus obtained

$$\int_0^\infty v(U-u)\,dy = U^2\delta^{**}\int_0^\infty \frac{v}{U}\left(1 - \frac{u}{U}\right)d\eta$$

$$= U^2\delta^{**}\frac{d\delta^{**}}{dx}\int_0^\infty\left(\eta\varphi - \int_0^\eta \varphi\,d\eta\right)(1-\varphi)\,d\eta -$$

$$UU'\delta^{**2}\int_0^\infty\left(\int_0^\eta \varphi\,d\eta\right)(1-\varphi)\,d\eta$$

or

$$\int_0^\infty v(U-u)\,dy = H_2U^2\,\delta^{**}\,\frac{d\delta^{**}}{dx} - H_3UU'\,\delta^{**\,2} \tag{2.7}$$

where H_2 and H_3 denote the constants

$$\left.\begin{aligned} H_2 &= \int_0^\infty\left(\eta\varphi - \int_0^\eta \varphi\,d\eta\right)(1-\varphi)\,d\eta \\ H_3 &= \int_0^\infty\left(\int_0^\eta \varphi\,d\eta\right)(1-\varphi)\,d\eta \end{aligned}\right\} \tag{2.8}$$

Finally, the last integral in equation (1.8) is transformed into

$$\int_0^\infty y(U-u)\,dy = H_4U\,\delta^{**2} \tag{2.9}$$

where the constant H_4 is equal to

$$H_4 = \int_0^\infty \eta(1-\varphi)\,d\eta \tag{2.10}$$

By substituting the integrals obtained in equation (1.8),

$$\frac{d}{dx}\left(H_1U^2\delta^{**2}\right) - H_2U^2\delta^{**}\frac{d\delta^{**}}{dx} + H_3UU'\delta^{**2} + H_4UU'\delta^{**2} = \nu U \tag{2.11}$$

or by replacing $\delta^{**2}/\nu = f/U'$ by equation (2.1) and carrying out the transformation,

$$\left(H_1 - \frac{1}{2}H_2\right)\frac{df}{dx} = \frac{U'}{U}\left[1 - \left(2H_1+H_3+H_4\right)f\right] + \frac{U''}{U'}\left(H_1 - \frac{1}{2}H_2\right)f \tag{2.12}$$

When the new constants are introduced,

$$\left.\begin{aligned} a &= \frac{1}{H_1 - \frac{1}{2}H_2} \\ b &= \frac{2H_1 + H_3 + H_4}{H_1 - \frac{1}{2}H_2} \end{aligned}\right\} \tag{2.13}$$

the equation of the first moment is reduced to the form

$$\frac{df}{dx} = \frac{U'}{U}(a-bf) + \frac{U''}{U'}f \tag{2.14}$$

The third equation is obtained from the system (1.9) by setting $k = 2$.

$$\frac{d}{dx}\int_0^\infty y^2u(U-u)dy - 2\int_0^\infty yv(U-u)dy + U'\int_0^\infty y^2(U-u)dy = 2\nu\int_0^\infty (U-u)dy \tag{2.15}$$

There is obtained, as before,

$$\int_0^\infty y^2u(U-u)dy = H_5U^2\delta^{**3} \tag{2.16}$$

where the constant H_5 is equal to

$$H_5 = \int_0^\infty \eta^2 \varphi(1-\varphi)d\eta \tag{2.17}$$

Further, by analogy with equation (2.7)

$$\int_0^\infty yv(U-u)dy = H_6 U^2 \delta^{**2} \frac{d\delta^{**}}{dx} - H_7 UU' \delta^{**3} \tag{2.18}$$

where

$$\left.\begin{aligned} H_6 &= \int_0^\infty \eta \left(\eta\,\varphi - \int_0^\eta \varphi\, d\eta\right)(1-\varphi)d\eta \\ H_7 &= \int_0^\infty \eta\left(\int_0^\eta \varphi\, d\eta\right)(1-\varphi)d\eta \end{aligned}\right\} \tag{2.19}$$

The last integral on the left side of equation (2.15) is equal to

$$\int_0^\infty y^2(U-u)dy = H_8 U\, \delta^{**3} \qquad \left(H_8 = \int_0^\infty \eta^2(1-\varphi)d\eta\right) \tag{2.20}$$

The integral in equation (2.15) on the right reduces to the unknown parameter H

$$\int_0^\infty (U-u)dy = U\int_0^\infty \left(1 - \frac{u}{U}\right) dy = U\delta^{**} \frac{\delta^*}{\delta^{**}} = U\, \delta^{**}H \tag{2.21}$$

By substituting the expressions obtained for the integrals in the second-moment equation (2.15), there is obtained, after simple transformations,

$$\frac{1}{4}\,(3H_5-2H_6)\,\frac{df}{dx} = \frac{U'}{U}\left[H - \left(H_5+H_7+\frac{1}{2}H_8\right) f\right] + \frac{1}{4}\,(3H_5-2H_6)\,\frac{U''}{U'}\, f \tag{2.22}$$

The system of three equations (2.3), (2.14), and (2.22) has thus been established for determining the three unknown magnitudes f, ζ, and H. The solution of this system is now considered.

3. Determination of the Constants H_i. Approximate Formulas for Parameters f, ζ, and H. For the determination of the numerical values of the constants $H_1, H_2, \ldots, H_8$, the form of the function $\varphi(\eta)$ must be known. The simplest velocity profile in the theory of the asymptotic boundary layer is the velocity profile in the sections of the boundary layer of the flow past a plate. The function $\varphi(\eta)$ for this case can easily be determined from the generally known table of values of the velocity ratio u/U as a function of $\xi = y\sqrt{U/\nu x}/2$.

Superfluous computations may be avoided by noting that the constants to be computed are connected with one another by certain simple relations.

First of all, from equations (2.3) and (2.14),

$$\zeta = \frac{1}{2} a + \left(2 + H - \frac{b}{2}\right) f \tag{3.1}$$

By setting $f = 0$, there is obtained $a = 2\zeta_0$, where ζ_0 denotes the magnitude ζ computed for the plate $(U' = 0,\ f = 0)$. From the definition of ζ and from the known relations for the plate,

$$a = 2\frac{\delta_0^{**}}{U}\frac{\tau_{w0}}{\mu} = 2\frac{0.664}{U}\sqrt{\frac{\nu x}{U}}\frac{0.332}{\mu}\sqrt{\frac{\mu\rho U^3}{x}} = 0.664^2 = 0.4408 \tag{3.2}$$

Further, by comparing with one another the magnitudes H_1, H_2, H_3, and H_4,

$$H_3 = H_1 - H_2 \tag{3.3}$$

$$H_4 - H_3 = \int_0^\infty \left(\eta - \int_0^\infty \varphi\, d\eta\right)(1-\varphi)\, d\eta = \frac{1}{2}\left[\int_0^\infty (1-\varphi)\, d\eta\right]^2 = \frac{1}{2} H_0^2 \tag{3.4}$$

where H_0 is the value of H for $f = 0$, that is, the ratio δ^*/δ^{**} for a plate is equal, as is known, to

$$H_0 = \frac{\delta^*}{\delta^{**}} = \frac{1.721}{0.664} = 2.59$$

It is then easy to obtain the value of b by equations (2.13), (3.3), and (3.4).

$$b = \frac{2H_1+H_3+H_4}{H_1 - \frac{1}{2} H_2} = a \left(2H_1+H_1-H_2+H_1-H_2+ \frac{1}{2}H_6{}^2\right)$$

$$= a \left(4H_1-2H_2+ \frac{1}{2}H_0{}^2\right) = 4 + \frac{a}{2} H_0{}^2 = 5.48 \approx 5.5 \qquad (3.5)$$

When df/dx is eliminated from equations (2.22) and (2.14),

$$H = \left(H_5+H_7+ \frac{1}{2}H_8\right) f + \frac{3H_5-2H_6}{4\left(H_1 - \frac{1}{2} H_2\right)} \left[1 - (2H_1+H_3+H_4)\ f\right]$$

$$= \left(H_5+H_7+ \frac{1}{2}H_8\right) f + \frac{a}{4} (3H_5-2H_6) \left(1 - \frac{b}{a} f\right) \qquad (3.6)$$

By setting $f = 0$,

$$\frac{1}{4} (3H_5-2H_6) = \frac{H_0}{a} = \frac{2.59}{0.44} \approx 5.89 \qquad (3.7)$$

The only magnitude that must be computed again from the table of values $\varphi(\eta)$ is the magnitude $H_5 + H_7 + H_8/2$. Numerical integration gives

$$H_5 + H_7 + \frac{1}{2} H_8 = 24.73 \qquad (3.8)$$

after which there is immediately obtained

$$H = 2.59 - 7.55 f \qquad (3.9)$$

Substituting this expression for H in equation (3.1) gives

$$\zeta = 0.22 + 1.85\ f - 7.55\ f^2 \tag{3.10}$$

Finally, integrating the simple linear equation (2.14) gives

$$f = \frac{aU'}{U^b}\int_0^x U^{b-1}(\xi)d\xi = \frac{0.44U'}{U^{5.5}}\int_0^x U^{4.5}(\xi)d\xi \tag{3.11}$$

Equations (3.9), (3.10) and (3.11) give the required solution.

The simple, approximate solution just obtained is now compared with the actual values. The almost complete agreement of the values of f obtained with the first approximation (which is practically the only one that is applied) of the preceding works (references 2 and 3) will be noted. The closed-form relation between ζ and f likewise differs little from the corresponding tabulated functions in the references cited.

For comparison, the curves $\zeta(f)$ and $H(f)$ obtained according to the formulas of reference 2 and by the formulas (3.10) and (3.9) are shown in figure 1. The results obtained will also be compared with the formulas of Wright and Bailey (reference 9). An approximate method of computation of the laminar boundary layer is proposed therein in which the equation of momentum (1.6) is employed with τ_w and δ^{**} substituted by the formulas for the flow past a plate. By expressing the results of Wright and Bailey in the parameters of the present report, the analogs of equations (3.9), (3.10), and (3.11) are obtained.

$$\left.\begin{aligned} H &= 2.59 \\ \zeta &= 0.22 + 4.09\ f \\ f &= 0.44\ \frac{U'x}{U} \end{aligned}\right\} \tag{3.12}$$

It is easily seen that this formula for f corresponds to equation (3.11) for $b = 1$. The straight lines for ζ and H shown dotted in figure 1 indicate the considerable deviation of the formulas of Wright and Bailey from more accurate formulas presented herein.

For confirmation, the particular case of the laminar boundary layer corresponding to the so-called single-slope velocity distribution at the outer boundary of the layer $U = 1 - x$ will be considered. This case has been theoretically solved and an exact solution in a tabulated form (reference 10) is available. The results of the recomputation of these accurate solutions in the form assumed by the parameters are given in figure 2. Also shown for comparison are the corresponding curves obtained by the proposed approximate method and by the method of Wright and Bailey.

4. Possible Methods of Rendering the Foregoing Solution More Accurate. The method described in the preceding sections was based on the assumption of a slight dependence of H_1 on the form parameters f, ζ, and H. This assumption may be eliminated and the method rendered more accurate, although it thereby becomes considerably more complicated.

In order to discuss the possible generalizations of the method, the complete system of equations, for example, for the three-parameter case is written out; that is, a three-parameter family of velocity profiles is assumed in place of equation (2.2).

$$\frac{u}{U} = \varphi(\eta;\ f,\ \zeta,\ H) \qquad \left(\eta = \frac{y}{\delta^{**}}\right) \tag{4.1}$$

By substituting this velocity profile in the system of the three equations of successive moments (1.6), (1.8) and (2.15), there is obtained a system of three ordinary nonlinear differential equations that determine the magnitudes of the parameters f, ζ, and H:

$$\frac{1}{2}\frac{U}{U'}\frac{df}{dx} + \left(2 - \frac{1}{2}\frac{UU''}{U'^2}\right) f + Hf = \zeta \tag{4.2}$$

$$\left[H_1 - \frac{1}{2} H_2 + \left(K_1 + \frac{\partial H_1}{\partial f}\right) f\right]\frac{df}{dx} + \left(K_2 + \frac{\partial H_1}{\partial \zeta}\right) f\ \frac{d\zeta}{dx} + \left(K_3 + \frac{\partial H_1}{\partial H}\right) f\ \frac{dH}{dx}$$

$$= \frac{U'}{U}\left[1 - (2H_1 + H_3 + H_4) f\right] + \frac{U''}{U'}\left(H_1 - \frac{1}{2} H_2\right) f \tag{4.3}$$

$$\left[\frac{1}{4}\left(3H_5-2H_6\right)+\left(K_4+\frac{1}{2}\frac{\partial H_5}{\partial f}\right)f\right]\frac{df}{dx}+\left(K_5+\frac{1}{2}\frac{\partial H_5}{\partial f}\right)f\frac{d\zeta}{dx}+$$

$$\left(K_6+\frac{1}{2}\frac{\partial H_5}{\partial H}\right)f\frac{dH}{dx}=\frac{U'}{U}\left[H-\left(H_5+H_7+\frac{1}{2}H_8\right)f\right]+$$

$$\frac{1}{4}\frac{U''}{U'}\left(3H_5-2H_6\right)f \tag{4.4}$$

in which, in addition to the previous notations, the following definitions are chosen:

$$\left.\begin{aligned}
K_1 &= \int_0^\infty\left(\int_0^\eta \frac{\partial\varphi}{\partial f}\,d\eta\right)(1-\varphi)d\eta\\
K_2 &= \int_0^\infty\left(\int_0^\eta \frac{\partial\varphi}{\partial\zeta}\,d\eta\right)(1-\varphi)d\eta\\
K_3 &= \int_0^\infty\left(\int_0^\eta \frac{\partial\varphi}{\partial H}\,d\eta\right)(1-\varphi)d\eta\\
K_4 &= \int_0^\infty \eta\left(\int_0^\eta \frac{\partial\varphi}{\partial f}\,d\eta\right)(1-\varphi)d\eta\\
K_5 &= \int_0^\infty \eta\left(\int_0^\eta \frac{\partial\varphi}{\partial\zeta}\,d\eta\right)(1-\varphi)d\eta\\
K_6 &= \int_0^\infty \eta\left(\int_0^\eta \frac{\partial\varphi}{\partial H}\,d\eta\right)(1-\varphi)d\eta
\end{aligned}\right\} \tag{4.5}$$

It is noted that, in the system of equations (4.2), (4.3), and (4.4), H_i and K_i are not constant magnitudes, as previously, but known functions of the form parameters f, ζ, and H; the form of these functions depends on the chosen family of profiles (4.1).

The equations (2.3), (2.14), and (2.22) earlier employed evidently represent a particular case of the system (4.2), (4.3), and (4.4) on the assumption that the family of velocity profiles at the different sections of the boundary layer has the form of equation (2.2); in other words, these profiles are similar to one another. All values of K_i are of course then equal to zero and H_i is constant.

The proposed method may be rendered considerably more accurate by assuming, for example, the single-parameter family of velocity profiles

$$\frac{u}{U} = \varphi(\eta;\ f) \tag{4.6}$$

Then

$$K_2 = \frac{\partial H_1}{\partial \zeta} = K_3 = \frac{\partial H_1}{\partial H} = K_5 = \frac{\partial H_5}{\partial \zeta} = K_6 = \frac{\partial H_5}{\partial H} = 0$$

and the system of equations (4.2), (4.3) and (4.4) is transformed as follows:

$$\frac{1}{2}\frac{U}{U'}\frac{df}{dx} + \left(2 - \frac{1}{2}\frac{UU''}{U'^2}\right) f + Hf = \zeta \tag{4.7}$$

$$\left[H_1 - \frac{1}{2} H_2 + \left(K_1 + \frac{\partial H_1}{\partial f}\right) f\right]\frac{df}{dx} = \frac{U'}{U}\left[1 - (2H_1 + H_3 + H_4)f\right] + \frac{U''}{U'}\left(H_1 - \frac{1}{2} H_2\right) f \tag{4.8}$$

$$\left[\frac{1}{4}(3H_5 - 2H_6) + \left(K_4 + \frac{1}{2}\frac{\partial H_5}{\partial f}\right) f\right]\frac{df}{dx} = \frac{U'}{U}\left[H - \left(H_5 + H_7 + \frac{1}{2} H_8\right) f\right] + \frac{1}{4}\frac{U''}{U'}(3H_5 - 2H_6) f \tag{4.9}$$

Equation (4.8) can be given the form

$$\frac{df}{dx} = \frac{U'}{U} \frac{1 - (2H_1+H_3+H_4)f}{H_1 - \frac{1}{2} H_2 + (K_1+\partial H_1/\partial f)f} + \frac{U''}{U'} \frac{H_1 - \frac{1}{2} H_2}{H_1 - \frac{1}{2} H_2 + (K_1+\partial H_1/\partial f)f} f \tag{4.10}$$

which represents a generalization of equation (2.12) where equation (4.10) approximates equation (2.12) because of the small change in H_1 with change in the parameter f and the smallness of the magnitude $(K_1+\partial H_1/\partial f)f$ in comparison with $H_1 - H_2/2$. This generalization permits obtaining the integral of equation (4.10) by introducing a correction to the solution of equation (2.12).

By dividing both sides of equation (4.9) by the corresponding sides of equation (4.8) and thus eliminating df/dx, there is obtained

$$H = \left(H_5+H_7+\frac{1}{2} H_8\right)f +$$

$$\frac{\frac{1}{4} (3H_5-2H_6) + \left(K_4+\frac{1}{2}\frac{\partial H_5}{\partial f}\right)f}{H_1 - \frac{1}{2} H_2 + \left(K_1+\frac{\partial H_1}{\partial f}\right)f} \left[1 - (2H_1+H_3+H_4)f\right] +$$

$$\frac{UU''}{U'^2} \left[\left(H_1 - \frac{1}{2} H_2\right) \frac{\frac{1}{4} (3H_5-2H_6) + \left(K_4+\frac{1}{2}\frac{\partial H_5}{\partial f}\right) f}{H_1 - \frac{1}{2} H_2 + \left(K_1 + \frac{\partial H_1}{\partial f}\right) f} - \frac{1}{4} (3H_5-2H_6)\right]f \tag{4.11}$$

By similar considerations on the smallness of the magnitudes $(K_4+1/2\ \partial H_5/\partial f)f$ in comparison with $(3H_5-2H_6)/4$ and of $(K_1+\partial H_1/\partial f)f$ in comparison with $H_1 - H_2/2$ and on the slight variability of H_i, it may be concluded that the value of H determined by equation (4.11) is an improvement in the accuracy of the approximate value of H according to equation (3.6).

It may be remarked that in this more accurate approximation there is no longer that universal relation between the parameters H and f, independent of the form of the function $U(x)$, characterizing the given particular problem. The presence in equation (4.11) of a second term with the factor UU''/U'^2 shows that in the more accurate approximation the magnitude H in a given section of the layer depends not only on the value of the form parameter f in this section, as was the case in the rougher approximation of equation (3.6) or (3.9), but also on the value of the magnitude UU''/U'^2 in the section considered, that is, on the values of the function $U(x)$ and its first two derivatives. It is readily observed that the second term on the right side of equation (4.11) will give a small correction to the solution (3.6) for relatively small values of the magnitude UU''/U'^2.

The same considerations hold for the expression for ζ, which may be obtained by substituting df/dx from equation (4.10) and H from equation (4.11) into equation (4.7):

$$\zeta = \frac{1 - (2H_1+H_3+H_4)f}{H_1 - \frac{1}{2}H_2 + \left(K_1+\frac{\partial H_1}{\partial f}\right)f}\left[\frac{1}{2} + \frac{1}{4}(3H_5-2H_6)f + \left(K_4 + \frac{1}{2}\frac{\partial H_5}{\partial f}\right)f^2\right] +$$

$$\frac{UU''}{U'^2}\,\frac{2f + \left(H_5+H_7+\frac{1}{2}H_8\right)f^2 - \left(K_1+\frac{\partial H_1}{\partial f}\right)f^2\left[\frac{1}{2} + (3H_5-2H_6)\frac{f}{4}\right] - \left(K_4 + \frac{1}{2}\frac{\partial H_5}{\partial f}\right)\left(H_1 - \frac{1}{2}H_2\right)f^3}{H_1 - \frac{1}{2}H_2 + \left(K_1+\frac{\partial H_1}{\partial f}\right)f} \tag{4.12}$$

As is seen, in this new approximation, in contrast to the preceding one, there is no universal relation between ζ and f. The presence of a term with the factor UU''/U'^2 makes the magnitude ζ depend not only on the value of the parameter f but also on the form of the function $U(x)$ and its first two derivatives in the given section of the boundary layer.

It is of interest to remark that in this approximation the position of the point of separation of the boundary layer, that is, the value of $x = x_s$ for which ζ is equal to zero, will no longer be determined by some universal value of the form parameter f_s, but in each individual case the value of $x = x_s$ must be determined for which the right side of equation (4.12) becomes zero.

By assuming a particular form of a family of velocity profiles (4.6), employing, for example, the sets of velocity profiles applied in the previous investigations (references 2 to 4), the values of the functions H_i and K_i are determined; the form parameters f, ζ, and H, that is, the thickness of the momentum loss δ^{**}, the friction stress τ_w and the displacement thickness δ^* may then be found without difficulty. The solution of equation (4.10) and the determination of H and ζ by equations (4.11) and (4.12) offers no particular difficulty. Further improvement in the accuracy requiring the solution of a system of the type of equations (4.2), (4.3) and (4.4) is hardly of practical interest.

In the previous discussion, the scheme of the asymptotically infinite boundary layer was used, but similar equations may be obtained also for the case where the boundary layer is assumed to be of finite thickness.

The method here proposed may evidently also be applied to the case of the thermal boundary layer. The characteristic feature of the method for the cases of both the dynamic and the thermal boundary layer lies in the fact that the friction stress and the quantity of heat given off by a unit area of the body are expressed in integral form and not in terms of the derivatives of functions that represent the approximate velocity and temperature distributions in the sections of the boundary layer.

Translated by S. Reiss
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REFERENCES

1. Loitsianskii, L. G.: Aerodynamics of the Boundary Layer. GTTI, 1941, pp. 170, 187.

2. Loitsianskii, L. G.: Approximate Method for Calculating the Laminar Boundary Layer on the Airfoil. Comptes Rendus (Doklady) de l'Acad. des Sci. de L'URSS, vol. XXXV, no. 8, 1942, pp. 227-232.

3. Kochin, N. E., and Loitsianskii, L. G.: An Approximate Method of Computation of the Boundary Layer. Doklady AN SSSR, T. XXXVI, No. 9, 1942.

4. Melnikov, A. P.: On Certain Problems in the Theory of a Wing in a Nonideal Medium. Doctoral dissertation, L. Voenno-vozdushnaia inzhenernaia akademia, 1942.

5. Leibenson, L. S.: Energetic Form of the Integral Condition in the Theory of the Boundary Layer. Rep. No. 240, CAHI, 1935.

6. Kochin, N. E., Kibel, I. A., and Roze, N. V.: Theoretical Hydrodynamics, pt. II. GTTI, 3d ed., 1948, p. 450.

7. Sutton, W. G. L.: An Approximate Solution of the Boundary Layer Equations for a Flat Plate. Phil. Mag. and Jour. Sci., vol. 23, ser. 7, 1937, pp. 1146-1152.

8. Carslaw, H., and Jaeger, J.: Operational Methods in Applied Mathematics. 1948.

9. Wright, E. A., and Bailey, G. W.: Laminar Frictional Resistance with Pressure Gradient. Jour. Aero. Sci., vol. 6, no. 12, Oct. 1939, pp. 485-488.

10. Howarth, L.: On the Solution of the Laminar Boundary Layer Equations. Proc. Roy. Soc. (London), vol. 164, no. A919, Feb. 1938, pp. 547-579.

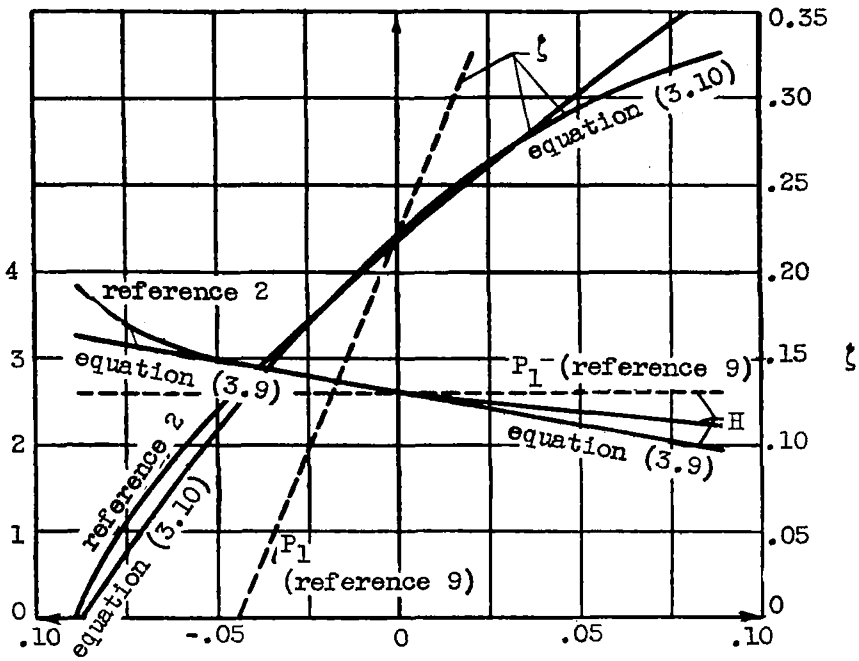

Figure 1.

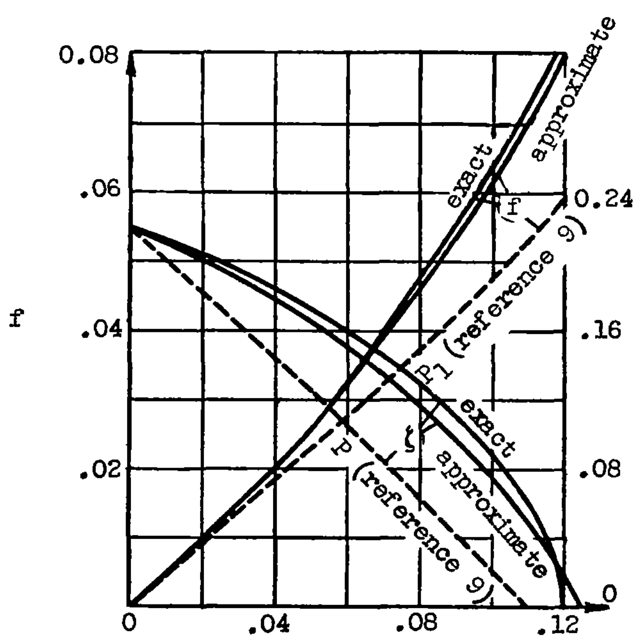

Figure 2.